Initiation to mould calculus through the example of saddle-node singularities

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Institut de mécanique céleste, CNRS
77 av. Denfert-Rochereau, 75014 Paris, France
email: sauzin@imcce.fr

Abstract. This article proposes an initiation to Écalle's mould calculus, a powerful combinatorial tool which yields surprisingly explicit formulas for the normalising series attached to an analytic germ of singular vector field. This is illustrated on the case of saddle-node singularities, generated by two-dimensional vector fields which are formally conjugate to Euler's vector field $x^2\frac{\partial}{\partial x}+(x+y)\frac{\partial}{\partial y}$, and for which the formal normalisation proves to be resurgent in $1/x$.

0 Introduction

This article is a survey of a part of a longer article [8] which aims at presenting in a systematic way some aspects of Écalle's theory of moulds, with the example of the classification of saddle-node singularities as a red thread.

Mould calculus was developed by J. Écalle in relation with his Resurgence theory almost thirty years ago [1] and the application to the saddle-node problem was indicated in [2] in concise manner.

Here, we omit much of the material of [8] and present the arguments in a different order, trying to explain the formal part of Écalle's method in fewer pages and hoping to arouse the reader's interest by this illustration of mould calculus.

1 Saddle-node singularities

Germs of holomorphic singular foliations of $(\mathbb{C}^2, 0)$ are defined by analytic differential equations of the form $P(x, y)\mathrm{d}y - Q(x, y)\mathrm{d}x = 0$ with P and $Q \in \mathbb{C}\{x, y\}$ both vanishing at the origin. Classifying such singular foliations means describing the conjugacy classes under the action of the group of germs of analytic invertible transformations of $(\mathbb{C}^2, 0)$; this is equivalent to classifying the corresponding singular vector fields $P(x, y)\frac{\partial}{\partial x} + Q(x, y)\frac{\partial}{\partial y}$ up to time-change.

We shall examine the case in which the foliation is assumed to be formally conjugate to the standard saddle-node foliation, defined by $x^2\mathrm{d}y - y\mathrm{d}x = 0$. This is the simplest case for differential equations whose 1-jets admit 0 and 1 as eigenvalues (the possible formal normal forms are $x^{p+1}\mathrm{d}y - (1+\lambda x)\mathrm{d}x = 0$, with $p \in \mathbb{N}^*$ and $\lambda \in \mathbb{C}$, and $y\,\mathrm{d}x = 0$).

In this case, by a classical theorem of Dulac ([5], [6]), the foliation can be analytically reduced to the form $x^2\mathrm{d}y - A(x,y)\mathrm{d}x = 0$, with

$$A(x,y) \in \mathbb{C}\{x,y\}, \quad A(0,y) = y, \quad \frac{\partial^2 A}{\partial x \partial y}(0,0) = 0. \tag{1}$$

Moreover, the vector fields corresponding to the foliation and to the normal form,

$$X = x^2 \frac{\partial}{\partial x} + A(x,y)\frac{\partial}{\partial y} \quad \text{and} \quad X_0 = x^2 \frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, \tag{2}$$

are themselves formally conjugate: there is a unique formal transformation of the form

$$\theta(x,y) = \big(x, \varphi(x,y)\big), \qquad \varphi(x,y) = y + \sum_{n\geq 0} \varphi_n(x) y^n, \qquad \varphi_n(x) \in x\mathbb{C}[[x]]. \tag{3}$$

such that

$$X = \theta^* X_0 \tag{4}$$

(no time-change is needed here). We call it the *formal normalisation* of the foliation (or of X itself).

Thus, in this case, our problem of analytic classification boils down to describing the analytic conjugacy classes of vector fields of the form X under the action of the group of "fibred" transformations (leaving x unchanged), knowing that they are all formally conjugate of one another (since all of them are formally conjugate to X_0).

Our starting point will be the data (1)–(2). We shall study the formal conjugacies (3) by means of Écalle's "moulds" and see how this leads to the "resurgent" character of the formal series which appear. (A similar study could be performed for the more general normal forms, with any p and λ.) We shall not give here the complete resurgent solution of the problem of analytic classification; the reader is referred to [8] for this and for the comparison with Martinet-Ramis's solution [5].

2 Formal separatrix, formal integral

Observe that our foliations always have an analytic separatrix (a leaf passing through the origin), namely the curve $\{x = 0\}$ in Dulac coordinates. For the vector field X, this corresponds to the solution $z \mapsto (0, u\mathrm{e}^z)$ (with an arbitrary constant of integration $u \in \mathbb{C}$).

The "formal curve" $\{y = \varphi_0(x)\}$ is to be considered as a "formal separatrix" of our foliation (and as a formal centre manifold of X), since it is the image by θ of $\{y = 0\}$ which is the other separatrix of the normal form. A time-parametrisation of the corresponding integral curve of X_0 is $z \mapsto (x,y) = (-1/z, 0)$. The formal series φ_0 can thus be obtained by setting $\widetilde{\varphi}_0(z) = \varphi_0(-1/z)$ and looking for a formal solution $z \mapsto \big(-1/z, \widetilde{\varphi}_0(z)\big)$ of X, i.e. the formal series $\widetilde{\varphi}_0$ must solve the non-linear differential equation

$$\frac{\mathrm{d}\widetilde{Y}}{\mathrm{d}z} = A(-1/z, \widetilde{Y}). \tag{5}$$

More generally, if we set

$$\widetilde{Y}(z,u) = u\,\mathrm{e}^z + \sum_{n\geq 0} u^n \mathrm{e}^{nz} \widetilde{\varphi}_n(z), \qquad \widetilde{\varphi}_n(z) = \varphi_n(-1/z) \in z^{-1}\mathbb{C}[[z^{-1}]], \tag{6}$$

we get what Écalle calls a "formal integral" of X, a formal object containing a free parameter and solving (5) (this is equivalent to finding a formal transformation of the form (3) which solves (4)). One can find the formal series $\widetilde{\varphi}_n$ by solving the ordinary differential equations obtained by expanding (5) in powers of u.

The simple and famous example of Euler's equation, for which $A(x,y) = x+y$, shows that the above formal series can be divergent. Indeed, the equation for $\varphi_0(x)$ is then $x^2 \frac{\mathrm{d}\varphi_0}{\mathrm{d}x} = x + \varphi_0$ and one finds $\varphi_0(x) = -\sum_{n\geq 1}(n-1)!x^n$. Since the equation is affine, there are no other non-trivial series in this case: $\varphi(x,y)$ boils down to $y + \varphi_0(x)$.

It is not by studying the differential equation (5) that we shall get information on the formal transformation θ, but rather by directly working on the conjugacy equation (3).

3 The formal normalisation as an operator

Let $\mathscr{A} = \mathbb{C}[[x,y]]$ and let ν denote the standard valuation (thus $\nu(f) \in \mathbb{N}$ is the "order" of f, with the convention that $\nu(x^m y^n) = m+n$). We denote by $\mathfrak{M}$ the maximal ideal of $\mathscr{A}$, consisting of all formal series without constant term, i.e. $\mathfrak{M} = \{ f \in \mathscr{A} \mid \nu(f) \in \mathbb{N}^* \}$.

We are given analytic vector fields X and X_0 as in (2), which are operators of the $\mathbb{C}$-algebra $\mathscr{A}$, more precisely $\mathbb{C}$-derivations (in fact, they are derivations of $\mathbb{C}\{x,y\}$, but we begin by forgetting analyticity). We shall look for the formal normalisation θ through its "substitution operator", which is the operator $\Theta \in \operatorname{End}_{\mathbb{C}} \mathscr{A}$ defined by

$$\Theta f = f \circ \theta, \qquad f \in \mathscr{A}. \tag{7}$$

There is in fact a one-to-one correspondence beween "substituable" pairs of formal series, i.e. pairs $\theta = (\theta_1, \theta_2) \in \mathfrak{M} \times \mathfrak{M}$, and operators of $\mathscr{A}$ which are *formally continuous algebra homomorphisms*, i.e. operators $\Theta \in \operatorname{End}_{\mathbb{C}} \mathscr{A}$ such that $\Theta(fg) = (\Theta f)(\Theta g)$ for any $f, g \in \mathscr{A}$ and $\nu(\Theta f_n) \to \infty$ for any sequence (f_n) of $\mathscr{A}$ such that $\nu(f_n) \to \infty$ (the proof is easy, see e.g. [8]; one goes from Θ to θ simply by setting $\theta_1 = \Theta x$ and $\theta_2 = \Theta y$).

Equation (4) can be written

$$(Xf) \circ \theta = X_0(f \circ \theta), \qquad f \in \mathscr{A},$$

and thus rephrased in terms of the substitution operator Θ as

$$\Theta X = X_0 \Theta. \tag{8}$$

Thus, looking for a formal invertible transformation solution of the conjugacy equation (4) *is equivalent to looking for a formally continuous algebra automorphism solution of* (8).

4 The formal normalisation as a mould expansion

The general strategy for finding a normalising operator by mould calculus consists in constructing it from the "building blocks" of the object one wishes to normalise. This implies that we shall restrict our attention to those operators of $\mathscr{A}$ which are obtained by combining the homogeneous components of X in all possible ways…

Since we are interested in fibred transformations, it is relevant to consider the homogeneous components of X relatively to the variable y only and to view $\mathbf{A} = \mathbb{C}[[x]]$ as a ring of scalars; we thus write

$$X = X_0 + \sum_{n\in\mathcal{N}} a_n B_n, \qquad B_n = y^{n+1}\frac{\partial}{\partial y}, \tag{9}$$

with coefficients $a_n \in \mathbf{A}$ stemming from the Taylor expansion

$$A(x,y) = y + \sum_{n\in\mathcal{N}} a_n(x) y^{n+1}, \qquad \mathcal{N} = \{\, n\in\mathbb{Z} \mid n \geq -1 \,\}. \tag{10}$$

Homogeneity here means that each $a_n B_n$ sends $\mathbf{A}y^k$ in $\mathbf{A}y^{k+n}$ for all $k\in\mathbb{N}$: the component $a_n B_n$ is homogeneous of degree n (this was the reason for shifting the index n by one unit in the Taylor expansion), while X_0 is homogeneous of degree 0.

We shall look for a solution of (8) *among all the operators of the form*

$$\Theta = \sum_{r\geq 0}\ \sum_{n_1,\dots,n_r\in\mathcal{N}} \mathcal{V}^{n_1,\dots,n_r} B_{n_r}\cdots B_{n_1}. \tag{11}$$

Here $(\mathcal{V}^{n_1,\dots,n_r})$ *is a collection of coefficients in* $\mathbf{A}$*, to be chosen in such a way that formula* (11) *has a meaning as a formally continuous operator of* $\mathscr{A}$ *and defines an automorphism solving* (8)*.* The above summation is better understood as a summation over $\mathcal{N}^\bullet$, the free monoid consisting of all words $\boldsymbol{\omega}$ (of any length r) the letters of which are taken in the alphabet $\mathcal{N}$. We thus set

$$\mathbf{B}_{\boldsymbol{\omega}} = B_{n_r}\cdots B_{n_1}, \qquad \boldsymbol{\omega} = (n_1,\dots,n_r)\in\mathcal{N}^\bullet, \tag{12}$$

and $\mathbf{B}_\emptyset = \mathrm{Id}$ for the only word of zero length ($\boldsymbol{\omega} = \emptyset$), and rewrite formula (11) as

$$\Theta = \sum_{\boldsymbol{\omega}\in\mathcal{N}^\bullet} \mathcal{V}^{\boldsymbol{\omega}} \mathbf{B}_{\boldsymbol{\omega}}. \tag{13}$$

An operator Θ defined by such a formula is called a mould expansion, or a mould-comould contraction. Here the *comould* is the map

$$\boldsymbol{\omega}\in\mathcal{N}^\bullet \mapsto \mathbf{B}_{\boldsymbol{\omega}} \in \mathrm{End}_{\mathbb{C}}\,\mathscr{A},$$

that we decided to define from the homogeneous components of X, and the *mould* is the map

$$\boldsymbol{\omega}\in\mathcal{N}^\bullet \mapsto \mathcal{V}^{\boldsymbol{\omega}} \in \mathbf{A},$$

that we must find so as to satisfy the aforementioned requirements.

5 The general framework for mould-comould contractions

The general theory of moulds and comoulds requires:

– an alphabet $\mathcal{N}$, which is simply a non-empty set, often with a structure of commutative semigroup (since it appears in practice as a set of possible degrees of homogeneity);

– a commutative $\mathbb{C}$-algebra $\mathbf{A}$ (the unit of which we denote by 1), in which moulds take their values;

– an $\mathbf{A}$-algebra $\mathscr{F}$ (the unit of which we denote by Id), possibly non-commutative, in which comoulds take their values. We also assume that a *complete ring pseudovaluation* $\operatorname{val}\colon \mathscr{F} \to \mathbb{Z} \cup \{\infty\}$ is given.[1]

In the saddle-node case we can choose for $\mathscr{F}$ a certain $\mathbf{A}$-subalgebra of $\operatorname{End}_{\mathbf{A}} \mathscr{A}$, with $\mathbf{A} = \mathbb{C}[[x]]$ and $\mathscr{A} = \mathbf{A}[[y]] = \mathbb{C}[[x,y]]$ (the fact that we deal with operators which commute with the multiplication by an element of $\mathbf{A}$ reflects the fibred character over x of the situation), defined as the set $\mathscr{F}_{\mathbf{A},\nu}$ of operators admitting a valuation with respect to the valuation ν of $\mathscr{A}$, i.e. the set of all $\Theta \in \operatorname{End}_{\mathbf{A}} \mathscr{A}$ for which there exists $\delta \in \mathbb{Z}$ such that $\nu(\Theta f) \geq \nu(f) + \delta$ for all $f \in \mathscr{A}$. This way, the valuation[2] ν of $\mathscr{A}$ induces a complete ring pseudovaluation val on $\mathscr{F}_{\mathbf{A},\nu}$, namely $\operatorname{val}(\Theta) = \inf_{f \in \mathscr{A} \setminus \{0\}} \{\nu(\Theta f) - \nu(f)\}$.

One can thus safely speak of "formally summable" families in $\mathscr{F}$: for instance, if for any $\delta \in \mathbb{Z}$ the set $\{\,\boldsymbol{\omega} \in \mathcal{N}^\bullet \mid \operatorname{val}(\mathcal{V}^{\boldsymbol{\omega}} \mathbf{B}_{\boldsymbol{\omega}}) \leq \delta\,\}$ is finite, then the family $(\mathcal{V}^{\boldsymbol{\omega}} \mathbf{B}_{\boldsymbol{\omega}})$ is formally summable and formula (13) defines an element of $\mathscr{F}$. We repeat our definitions in this general context:

– a mould is any map $\mathcal{N}^\bullet \to \mathbf{A}$ (we usually denote by $M^\bullet$ the mould whose value on the word $\boldsymbol{\omega}$ is $M^{\boldsymbol{\omega}}$),

– a comould is any map $\mathcal{N}^\bullet \to \mathscr{F}$ (we usually denote by $\mathbf{B}_\bullet$ the comould whose value on the word $\boldsymbol{\omega}$ is $\mathbf{B}_{\boldsymbol{\omega}}$),

– a mould expansion is the result of the contraction of a mould $M^\bullet$ and a comould $\mathbf{B}_\bullet$ such that the family $(M^{\boldsymbol{\omega}} \mathbf{B}_{\boldsymbol{\omega}})_{\boldsymbol{\omega} \in \mathcal{N}^\bullet}$ is formally summable in $\mathscr{F}$; we usually use the short-hand notation
$$\Theta = \sum M^\bullet \mathbf{B}_\bullet.$$

The monoid law in $\mathcal{N}^\bullet$ is the concatenation, denoted by $\centerdot$, which allows us to define *mould multiplication* by the formula
$$P^\bullet = M^\bullet \times N^\bullet \colon\ \boldsymbol{\omega} \mapsto P^{\boldsymbol{\omega}} = \sum_{\boldsymbol{\omega} = \boldsymbol{\omega}^1 \centerdot \boldsymbol{\omega}^2} M^{\boldsymbol{\omega}^1} N^{\boldsymbol{\omega}^2}.$$

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[1] This means that $\operatorname{val}(\Theta) = \infty \iff \Theta = 0$, $\operatorname{val}(\Theta_1 - \Theta_2) \geq \min\{\operatorname{val}(\Theta_1), \operatorname{val}(\Theta_2)\}$, $\operatorname{val}(\Theta_1\Theta_2) \geq \operatorname{val}(\Theta_1) + \operatorname{val}(\Theta_2)$ and the distance $(\Theta_1, \Theta_2) \mapsto 2^{-\operatorname{val}(\Theta_2 - \Theta_1)}$ is complete.

[2] For technical reasons, rather than the standard valuation on $\mathbb{C}[[x,y]]$, we shall use another monomial valuation, defined by $\nu(x^m y^n) = 4m + n$ (see Section 6).

The space of moulds is in fact an $\mathbf{A}$-algebra. Correspondingly, if a comould $\mathbf{B}_\bullet$ is *mutiplicative*, in the sense that $\mathbf{B}_{\boldsymbol{\omega}^1\bullet\boldsymbol{\omega}^2} = \mathbf{B}_{\boldsymbol{\omega}^2}\mathbf{B}_{\boldsymbol{\omega}^1}$ for any $\boldsymbol{\omega}^1, \boldsymbol{\omega}^2 \in \mathcal{N}^\bullet$ (as is obviously the case for a comould defined as in (12)), then

$$\sum (M^\bullet \times N^\bullet)\, \mathbf{B}_\bullet = \Big(\sum N^\bullet \mathbf{B}_\bullet\Big)\Big(\sum M^\bullet \mathbf{B}_\bullet\Big)$$

as soon as both expressions in the right-hand side are formally summable.

One can easily check that a mould $M^\bullet$ has a multiplicative inverse if and only if $M^\emptyset$ is invertible in $\mathbf{A}$. In this case, if $(M^{\boldsymbol{\omega}}\mathbf{B}_{\boldsymbol{\omega}})$ is formally summable and $\mathbf{B}_\bullet$ is multiplicative, then $\sum M^\bullet \mathbf{B}_{\boldsymbol{\omega}}$ is invertible in $\mathscr{F}$.

We do not develop farther the theory here and prefer to return to the formal normalisation of our saddle-node singularity X.

6 Solution of the formal conjugacy problem

Let us use the $\mathbf{A}$-algebra $\mathscr{F} = \mathscr{F}_{\mathbf{A},\nu}$ defined in the previous section, to which each operator $B_n = y^{n+1}\partial_y$ obviously belongs (B_n is $\mathbf{A}$-linear and $\operatorname{val}(B_n) = n$), and thus also the $\mathbf{B}_{\boldsymbol{\omega}}$'s defined by (12). Formula (9) can be written $X - X_0 = \sum J_a^\bullet \mathbf{B}_\bullet$, with

$$J_a^{\boldsymbol{\omega}} = \left|\begin{array}{ll} a_{n_1} & \text{if } \boldsymbol{\omega} = (n_1) \\ 0 & \text{if } r(\boldsymbol{\omega}) \neq 1 \end{array}\right. \tag{14}$$

and the conjugacy equation (8) is equivalent to

$$\Theta(X - X_0) = [X_0, \Theta]. \tag{15}$$

Lemma 6.1. *Let $\mathcal{V}^\bullet$ be a mould such that the family $(\mathcal{V}^{\boldsymbol{\omega}}\mathbf{B}_{\boldsymbol{\omega}})$ is formally summable in $\mathscr{F}$. Then*

$$[X_0, \sum \mathcal{V}^\bullet \mathbf{B}_\bullet] = \sum (x^2\partial_x \mathcal{V}^\bullet + \nabla\mathcal{V}^\bullet)\mathbf{B}_\bullet,$$

where the mould $\nabla\mathcal{V}^\bullet$ is defined by $\nabla\mathcal{V}^\emptyset = 0$ and

$$\nabla\mathcal{V}^{\boldsymbol{\omega}} = (n_1 + \cdots + n_r)\mathcal{V}^{\boldsymbol{\omega}}$$

for $\boldsymbol{\omega} = (n_1, \ldots, n_r)$ non-empty.

Proof. The operator $\mathbf{B}_{n_1,\ldots,n_r}$ is homogeneous of degree $n_1 + \cdots + n_r$. One can check that, if $\Theta \in \operatorname{End}_{\mathbf{A}}(\mathbf{A}[[y]])$ is homogeneous of degree $n \in \mathbb{Z}$, then

$$[y\partial_y, \Theta] = n\Theta.$$

Indeed, by $\mathbf{A}$-linearity and formal continuity, it is sufficient to check that both operators act the same way on a monomial y^k; but $\Theta y^k = \beta_k y^{k+n}$ with a $\beta_k \in \mathbf{A}$, thus $y\partial_y \Theta y^k = (k+n)\beta_k y^{k+n} = (k+n)\Theta y^k$ while $\Theta y \partial_y y^k = k\Theta y^k$.

Since $X_0 = x^2\partial_x + y\partial_y$ and $x^2\partial_x$ commutes with the B_n's, it follows that

$$[X_0, \mathcal{V}^{\boldsymbol{\omega}}\mathbf{B}_{\boldsymbol{\omega}}] = \big(x^2\partial_x\mathcal{V}^{\boldsymbol{\omega}} + (n_1 + \cdots + n_r)\mathcal{V}^{\boldsymbol{\omega}}\big)\mathbf{B}_{\boldsymbol{\omega}}, \qquad \boldsymbol{\omega} = (n_1, \ldots, n_r) \in \mathcal{N}^\bullet.$$

The conlusion follows by formal continuity. $\square$

Looking for a solution of the form $\Theta = \sum \mathcal{V}^\bullet \mathbf{B}_\bullet$ for equation (15), we are thus led to the mould equation

$$x^2\partial_x \mathcal{V}^\bullet + \nabla \mathcal{V}^\bullet = J_a^\bullet \times \mathcal{V}^\bullet. \tag{16}$$

Lemma 6.2. *Equation* (16) *has a unique solution* $\mathcal{V}^\bullet$ *such that* $\mathcal{V}^\emptyset = 1$ *and* $\mathcal{V}^{\boldsymbol{\omega}} \in x\mathbb{C}[[x]]$ *for every non-empty* $\boldsymbol{\omega} \in \mathcal{N}^\bullet$. *Moreover,*

$$\mathcal{V}^{n_1,\dots,n_r} \in x^{\lceil r/2 \rceil}\mathbb{C}[[x]], \tag{17}$$

where $\lceil s \rceil$ *denotes, for any* $s \in \mathbb{R}$, *the least integer not smaller than* s.

Proof. Let us perform the change of variable $z = -1/x$ and set $\partial = \frac{\mathrm{d}}{\mathrm{d}z}$ and $\widetilde{a}_n(z) = a_n(-1/z)$. Observe that

$$\widetilde{a}_n \in z^{-1}\mathbb{C}[[z^{-1}]], \qquad \widetilde{a}_0 \in z^{-2}\mathbb{C}[[z^{-1}]], \tag{18}$$

as a consequence of (1).

The equation for $\widetilde{\mathcal{V}}^{\boldsymbol{\omega}}(z) = \mathcal{V}^{\boldsymbol{\omega}}(-1/z)$, with $\boldsymbol{\omega} = (n_1,\dots,n_r)$, $r \geq 1$, is

$$(\partial + n_1 + \cdots + n_r)\widetilde{\mathcal{V}}^{n_1,\dots,n_r} = \widetilde{a}_{n_1}\widetilde{\mathcal{V}}^{n_2,\dots,n_r}. \tag{19}$$

On the one hand, $\partial + \mu$ is an invertible operator of $\mathbb{C}[[z^{-1}]]$ for any $\mu \in \mathbb{C}^*$ and the inverse operator

$$(\partial + \mu)^{-1} = \sum_{r \geq 0} \mu^{-r-1}(-\partial)^r \tag{20}$$

leaves $z^{-1}\mathbb{C}[[z^{-1}]]$ invariant; on the other hand, when $\mu = 0$, ∂ induces an isomorphism $z^{-2}\mathbb{C}[[z^{-1}]] \to z^{-1}\mathbb{C}[[z^{-1}]]$.

For $r = 1$, equation (19) has a unique solution in $z^{-1}\mathbb{C}[[z^{-1}]]$, because the right-hand side is $\widetilde{a}_{n_1}$, element of $z^{-1}\mathbb{C}[[z^{-1}]]$, and even of $z^{-2}\mathbb{C}[[z^{-1}]]$ when $n_1 = 0$. By induction, for $r \geq 2$, we get a right-hand side in $z^{-2}\mathbb{C}[[z^{-1}]]$ and a unique solution $\widetilde{\mathcal{V}}^{\boldsymbol{\omega}}$ in $z^{-1}\mathbb{C}[[z^{-1}]]$ for $\boldsymbol{\omega} = (n_1,\dots,n_r) \in \mathcal{N}^r$. Moreover, with the notation ${}^{\text{‘}}\boldsymbol{\omega} = (n_2,\dots,n_r)$, we have

$$v(\widetilde{\mathcal{V}}^{\boldsymbol{\omega}}) \geq \alpha^{\boldsymbol{\omega}} + v(\widetilde{\mathcal{V}}^{{}^{\text{‘}}\boldsymbol{\omega}}), \qquad \text{with } \alpha^{\boldsymbol{\omega}} = \left| \begin{array}{ll} 0 & \text{if } n_1 + \cdots + n_r = 0 \text{ and } n_1 \neq 0, \\ 1 & \text{if } n_1 + \cdots + n_r \neq 0 \text{ or } n_1 = 0, \end{array} \right.$$

where v denotes the standard valuation of $\mathbb{C}[[z^{-1}]]$. Thus $v(\widetilde{\mathcal{V}}^{\boldsymbol{\omega}}) \geq \operatorname{card} \mathcal{R}^{\boldsymbol{\omega}}$, with $\mathcal{R}^{\boldsymbol{\omega}} = \{\, i \in [1,r] \mid n_i + \cdots + n_r \neq 0 \text{ or } n_i = 0 \,\}$ for $r \geq 1$.

Let us check that $\operatorname{card} \mathcal{R}^{\boldsymbol{\omega}} \geq \lceil r/2 \rceil$. This stems from the fact that if $i \notin \mathcal{R}^{\boldsymbol{\omega}}$, $i \geq 2$, then $i - 1 \in \mathcal{R}^{\boldsymbol{\omega}}$ (indeed, in that case $n_{i-1} + \cdots + n_r = n_{i-1}$), and that $\mathcal{R}^{\boldsymbol{\omega}}$ has at least one element, namely r. The inequality is thus true for $r = 1$ or 2; by induction, if $r \geq 3$, then $\mathcal{R}^{\boldsymbol{\omega}} \cap [3,r] = \mathcal{R}^{{}^{\text{“}}\boldsymbol{\omega}}$ with ${}^{\text{“}}\boldsymbol{\omega} = (n_3,\dots,n_r)$ and either $2 \in \mathcal{R}^{\boldsymbol{\omega}}$, or $2 \notin \mathcal{R}^{\boldsymbol{\omega}}$ and $1 \in \mathcal{R}^{\boldsymbol{\omega}}$, thus $\operatorname{card} \mathcal{R}^{\boldsymbol{\omega}} \geq 1 + \operatorname{card} \mathcal{R}^{{}^{\text{“}}\boldsymbol{\omega}}$. $\square$

The formal summability of the family $(\mathcal{V}^{\boldsymbol{\omega}}\mathbf{B}_{\boldsymbol{\omega}})$ follows from (17) if we use the modified valuation of footnote 2. Indeed, one gets $\operatorname{val}(\mathcal{V}^{n_1,\dots,n_r}\mathbf{B}_{n_1,\dots,n_r}) \geq n_1 + \cdots + n_r + 2r$. The n_i's may be negative but they are always ≥ -1, thus $n_1 + \cdots + n_r + r \geq 0$. Therefore, for any $\delta > 0$, the condition $n_1 + \cdots + n_r + 2r \leq \delta$

implies $r \le \delta$ and $\sum(n_i+1) = n_1+\cdots+n_r+r \le \delta$. Since this condition is fulfilled only a finite number of times, the summability follows.

Setting $\Theta = \sum \mathcal{V}^\bullet \mathbf{B}_\bullet$*, we thus get a solution of the conjugacy equation* (8) *and* Θ *is a continuous operator of* $\mathscr{A}$*.*

But is Θ an algebra automorphism? If one takes for granted the existence of a unique θ of the form (3) which solves (4) (and this is not hard to check), then one can easily identify the operator Θ that we just defined with the substitution operator corresponding to θ and Θ is thus an algebra automorphism. But it is possible to prove directly this fact by checking a certain symmetry property of the mould $\mathcal{V}^\bullet$, called *symmetrality*.

7 Cosymmetrality and symmetrality

Let us return for a while to the general context of Section 5, with an alphabet $\mathcal{N}$ and a commutative $\mathbb{C}$-algebra $\mathbf{A}$, focusing on the case where $\mathbf{B}_\bullet$ is the multiplicative comould generated by a family of $\mathbf{A}$-linear derivations $(B_n)_{n\in\mathcal{N}}$ of a commutative algebra $\mathscr{A}$.

We thus assume that $\mathscr{A}$ is a commutative $\mathbf{A}$-algebra, on which a complete ring pseudovaluation ν is given, that $\mathscr{F} = \mathscr{F}_{\mathbf{A},\nu}$ and that, for each $n \in \mathcal{N}$, we are given $B_n \in \mathscr{F}$ satisfying the Leibniz rule

$$B_n(fg) = (B_n f)g + f(B_n g), \qquad f, g \in \mathscr{A}.$$

This property can be rewritten

$$\sigma(B_n) = B_n \otimes \mathrm{Id} + \mathrm{Id} \otimes B_n,$$

with the notation $\sigma\colon \operatorname{End}_{\mathbf{A}} \mathscr{A} \to \operatorname{Bil}_{\mathbf{A}}(\mathscr{A}\times\mathscr{A},\mathscr{A})$ for the composition with the multiplication of $\mathscr{A}$ and $\Theta_1 \otimes \Theta_2(f,g) := (\Theta_1 f)(\Theta_2 g)$ for any $\Theta_1, \Theta_2 \in \operatorname{End}_{\mathbf{A}} \mathscr{A}$.

We now consider the comould defined by

$$\mathbf{B}_\emptyset = \mathrm{Id}, \qquad \mathbf{B}_{\boldsymbol{\omega}} = B_{n_r}\cdots B_{n_1} \quad \text{for non-empty } \boldsymbol{\omega} = (n_1,\dots,n_r) \in \mathcal{N}^\bullet.$$

One can easily check, by iteration of the Leibniz rule, that

$$\sigma\big(\mathbf{B}_{(n_1,n_2)}\big) = \mathbf{B}_{(n_1,n_2)} \otimes \mathbf{B}_\emptyset + \mathbf{B}_{(n_1)} \otimes \mathbf{B}_{(n_2)} + \mathbf{B}_{(n_2)} \otimes \mathbf{B}_{(n_1)} + \mathbf{B}_\emptyset \otimes \mathbf{B}_{(n_1,n_2)},$$

$$\begin{aligned}\sigma\big(\mathbf{B}_{(n_1,n_2,n_3)}\big) = {} & \mathbf{B}_{(n_1,n_2,n_3)} \otimes \mathbf{B}_\emptyset \\ & + \mathbf{B}_{(n_1,n_2)} \otimes \mathbf{B}_{(n_3)} + \mathbf{B}_{(n_1,n_3)} \otimes \mathbf{B}_{(n_2)} + \mathbf{B}_{(n_2,n_3)} \otimes \mathbf{B}_{(n_1)} \\ & + \mathbf{B}_{(n_3)} \otimes \mathbf{B}_{(n_1,n_2)} + \mathbf{B}_{(n_2)} \otimes \mathbf{B}_{(n_1,n_3)} + \mathbf{B}_{(n_1)} \otimes \mathbf{B}_{(n_2,n_3)} \\ & + \mathbf{B}_\emptyset \otimes \mathbf{B}_{(n_1,n_2,n_3)},\end{aligned}$$

the general formula being

$$\sigma(\mathbf{B}_{\boldsymbol{\omega}}) = \sum_{\boldsymbol{\omega}^1,\boldsymbol{\omega}^2\in\mathcal{N}^\bullet} \operatorname{sh}\binom{\boldsymbol{\omega}^1,\,\boldsymbol{\omega}^2}{\boldsymbol{\omega}} \mathbf{B}_{\boldsymbol{\omega}^1} \otimes \mathbf{B}_{\boldsymbol{\omega}^2}, \tag{21}$$

where $\operatorname{sh}\binom{\boldsymbol{\omega}^1,\boldsymbol{\omega}^2}{\boldsymbol{\omega}}$ denotes the number of obtaining $\boldsymbol{\omega}$ by *shuffling* of $\boldsymbol{\omega}^1$ and $\boldsymbol{\omega}^2$: it is the number of permutations σ such that one can write $\boldsymbol{\omega}^1 = (\omega_1,\dots,\omega_\ell)$,

$\boldsymbol{\omega}^2 = (\omega_{\ell+1},\dots,\omega_r)$ and $\boldsymbol{\omega} = (\omega_{\sigma(1)},\dots,\omega_{\sigma(r)})$ with the property $\sigma(1) < \dots < \sigma(\ell)$ and $\sigma(\ell+1) < \dots < \sigma(r)$ (thus it is non-zero only if $\boldsymbol{\omega}$ can be obtained by interdigitating the letters of $\boldsymbol{\omega}^1$ and those of $\boldsymbol{\omega}^2$ while preserving their internal order in $\boldsymbol{\omega}^1$ or $\boldsymbol{\omega}^2$).

Any comould satisfying (21) is said to be *cosymmetral*.

Dually, one says that a mould $M^\bullet$ is *symmetral* if and only if $M^\emptyset = 1$ and, for any two non-empty words $\boldsymbol{\omega}^1, \boldsymbol{\omega}^2$,

$$\sum_{\boldsymbol{\omega}\in\mathcal{N}^\bullet} \operatorname{sh}\binom{\boldsymbol{\omega}^1,\,\boldsymbol{\omega}^2}{\boldsymbol{\omega}} M^{\boldsymbol{\omega}} = M^{\boldsymbol{\omega}^1} M^{\boldsymbol{\omega}^2}. \tag{22}$$

One says that a mould $M^\bullet$ is *alternal* if and only if $M^\emptyset = 0$ and, for any two non-empty words $\boldsymbol{\omega}^1, \boldsymbol{\omega}^2$, the above sum vanishes.

Suppose that $\mathbf{B}_\bullet$ is cosymmetral and that the family $(M^{\boldsymbol{\omega}}\mathbf{B}_{\boldsymbol{\omega}})$ is formally summable, with $\Theta = \sum M^\bullet \mathbf{B}_\bullet$. It is easy to check that

$$\begin{aligned} M^\bullet \text{ symmetral} \quad &\Rightarrow \quad \sigma(\Theta) = \Theta\otimes\Theta, \\ M^\bullet \text{ alternal} \quad &\Rightarrow \quad \sigma(\Theta) = \Theta\otimes\mathrm{Id} + \mathrm{Id}\otimes\Theta. \end{aligned}$$

In other words, contraction with a symmetral mould yields an automorphism, while contraction with an alternal mould yields a derivation.

Symmetral and alternal moulds satisfy many stability properties (see [8] §5). Here we just mention that the multiplicative inverse $\widetilde{M}^\bullet$ of a symmetral mould $M^\bullet$ is the symmetral mould defined by

$$\widetilde{M}^{n_1,\dots,n_r} = (-1)^r M^{n_r,\dots,n_1}. \tag{23}$$

Since the multiplicative comould generated by a family of derivations is cosymmetral, in the case of the saddle-node it is sufficient to check the following lemma to prove that the Θ defined in Section 6 is indeed an automorphism:

Lemma 7.1. *The mould $\mathcal{V}^\bullet$ defined by Lemma 6.2 is symmetral.*

Proof. We must show that

$$\mathcal{V}^{\boldsymbol{\alpha}}\mathcal{V}^{\boldsymbol{\beta}} = \sum_{\boldsymbol{\gamma}\in\mathcal{N}^\bullet} \operatorname{sh}\binom{\boldsymbol{\alpha},\,\boldsymbol{\beta}}{\boldsymbol{\gamma}} \mathcal{V}^{\boldsymbol{\gamma}}, \qquad \boldsymbol{\alpha},\boldsymbol{\beta}\in\mathcal{N}^\bullet. \tag{24}$$

Since $\mathcal{V}^\emptyset = 1$, this is obviously true for $\boldsymbol{\alpha}$ or $\boldsymbol{\beta} = \emptyset$. We now argue by induction on $r = r(\boldsymbol{\alpha}) + r(\boldsymbol{\beta})$. We thus suppose that $r \geq 1$ and, without loss of generality, both of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ non-empty. With the notations $d = x^2\frac{\mathrm{d}}{\mathrm{d}x}$, $\|\boldsymbol{\alpha}\| = \alpha_1 + \dots + \alpha_{r(\boldsymbol{\alpha})}$ and $\|\boldsymbol{\beta}\| = \beta_1 + \dots + \beta_{r(\boldsymbol{\beta})}$, we compute

$$\begin{aligned} A := (d + \|\boldsymbol{\alpha}\| + \|\boldsymbol{\beta}\|) \sum_{\boldsymbol{\gamma}} \operatorname{sh}\big(\tbinom{\boldsymbol{\alpha},\boldsymbol{\beta}}{\boldsymbol{\gamma}}\big)\mathcal{V}^{\boldsymbol{\gamma}} \\ = \sum_{\boldsymbol{\gamma}\neq\emptyset} \operatorname{sh}\big(\tbinom{\boldsymbol{\alpha},\boldsymbol{\beta}}{\boldsymbol{\gamma}}\big)(d + \|\boldsymbol{\gamma}\|)\mathcal{V}^{\boldsymbol{\gamma}} = \sum_{\boldsymbol{\gamma}\neq\emptyset} \operatorname{sh}\big(\tbinom{\boldsymbol{\alpha},\boldsymbol{\beta}}{\boldsymbol{\gamma}}\big) a_{\gamma_1}\mathcal{V}^{`\boldsymbol{\gamma}}, \end{aligned}$$

using the notations $\|\boldsymbol{\gamma}\| = \gamma_1 + \cdots + \gamma_s$ and ${}^{\backprime}\boldsymbol{\gamma} = (\gamma_2, \dots, \gamma_s)$ for any non-empty $\boldsymbol{\gamma} = (\gamma_1, \dots, \gamma_s)$ (with the help (16) for the last identity). Splitting the last summation according to the value of γ_1, which must be α_1 or β_1, we get

$$A = \sum_{\boldsymbol{\delta}} \operatorname{sh}\left({}^{{}^{\backprime}\boldsymbol{\alpha},\boldsymbol{\beta}}_{\;\;\boldsymbol{\delta}} \right) a_{\alpha_1} \mathcal{V}^{\boldsymbol{\delta}} + \sum_{\boldsymbol{\delta}} \operatorname{sh}\left({}^{\boldsymbol{\alpha},{}^{\backprime}\boldsymbol{\beta}}_{\;\;\boldsymbol{\delta}} \right) a_{\beta_1} \mathcal{V}^{\boldsymbol{\delta}} = a_{\alpha_1} \mathcal{V}^{{}^{\backprime}\boldsymbol{\alpha}} \cdot \mathcal{V}^{\boldsymbol{\beta}} + \mathcal{V}^{\boldsymbol{\alpha}} \cdot a_{\beta_1} \mathcal{V}^{{}^{\backprime}\boldsymbol{\beta}}$$

(using the induction hypothesis), hence, using again (16),

$$A = (d + \|\boldsymbol{\alpha}\|)\mathcal{V}^{\boldsymbol{\alpha}} \cdot \mathcal{V}^{\boldsymbol{\beta}} + \mathcal{V}^{\boldsymbol{\alpha}} \cdot (d + \|\boldsymbol{\beta}\|)\mathcal{V}^{\boldsymbol{\beta}} = (d + \|\boldsymbol{\alpha}\| + \|\boldsymbol{\beta}\|)(\mathcal{V}^{\boldsymbol{\alpha}}\mathcal{V}^{\boldsymbol{\beta}}).$$

We conclude that both sides of (24) must coincide, because $d + \|\boldsymbol{\alpha}\| + \|\boldsymbol{\beta}\|$ is invertible if $\|\boldsymbol{\alpha}\| + \|\boldsymbol{\beta}\| \neq 0$ and both of them belong to $x\mathbb{C}[[x]]$, thus even if $\|\boldsymbol{\alpha}\| + \|\boldsymbol{\beta}\| = 0$ the desired conclusion holds. □

8 Resurgence of the formal conjugacy

At this stage, we have found formal series $\mathcal{V}^{\boldsymbol{\omega}} \in \mathbb{C}[[x]]$ which determine a formally continuous algebra automorphism $\Theta = \sum \mathcal{V}^{\bullet}\mathbf{B}_{\bullet}$ conjugating X_0 and X. Since $\Theta x = x$, we deduce that Θ is the substitution operator associated with the formal transformation $\theta(x,y) = \big(x, \varphi(x,y)\big)$ where $\varphi = \Theta y$. This is what was announced in (3).

The components $\varphi_n(x)$ of $\varphi(x,y)$ are easily computed: on checks by induction that

$$\mathbf{B}_{\boldsymbol{\omega}} y = \beta_{\boldsymbol{\omega}} y^{n_1+\cdots+n_r+1}, \qquad \boldsymbol{\omega} = (n_1, \dots, n_r),\ r \geq 1, \tag{25}$$

with $\beta_{\boldsymbol{\omega}} = 1$ if $r = 1$, $\beta_{\boldsymbol{\omega}} = (n_1+1)(n_1+n_2+1)\cdots(n_1+\cdots+n_{r-1}+1)$ if $r \geq 2$; one has $\beta_{\boldsymbol{\omega}} = 0$ whenever $n_1 + \cdots + n_r \leq -2$ (since (25) holds a priori in the fraction field $\mathbb{C}(\!(y)\!)$ but $\mathbf{B}_{\boldsymbol{\omega}} y$ belongs to $\mathbb{C}[[y]]$), hence

$$\varphi(x,y) = \Theta y = y + \sum_{n \geq 0} \varphi_n(x) y^n, \qquad \varphi_n = \sum_{\substack{r \geq 1,\, \boldsymbol{\omega} \in \mathcal{N}^r \\ n_1+\cdots+n_r+1=n}} \beta_{\boldsymbol{\omega}} \mathcal{V}^{\boldsymbol{\omega}} \tag{26}$$

(in the series giving φ_n, there are only finitely many terms for each r, (17) thus yields its formal convergence in $x\mathbb{C}[[x]]$).

Similarly, if we define $\mathcal{V}\!\!\!/^{\bullet}$ as the multiplicative inverse of $\mathcal{V}^{\bullet}$ with the help of formula (23), then $\Theta^{-1} = \sum \mathcal{V}\!\!\!/^{\bullet}\mathbf{B}_{\bullet}$ is the substitution operator of a formal transformation $(x,y) \mapsto \big(x, \psi(x,y)\big)$, which is nothing but θ^{-1}, and

$$\psi(x,y) = \Theta^{-1} y = y + \sum_{n \geq 0} \psi_n(x) y^n, \tag{27}$$

where each coefficient can be represented as a formally convergent series $\psi_n = \sum_{n_1+\cdots+n_r+1=n} \beta_{\boldsymbol{\omega}} \mathcal{V}\!\!\!/^{\boldsymbol{\omega}}$.

These are remarkably explicit formulas, quite different from what one would have obtained by solving directly the differential equation (5) for $\widetilde{\varphi}_0(z) = \varphi_0(-1/z)$ for instance.

An advantage of these formulas is that they allow to prove the resurgent character with respect to the variable $z = -1/x$ of all the formal series which appear in our problem. We recall that a formal series

$$\widetilde{\varphi}(z) = \sum_{n\geq 0} c_n z^{-n-1} \in z^{-1}\mathbb{C}[[z^{-1}]]$$

is said to be resurgent if its formal Borel transform

$$\widehat{\varphi}(\zeta) = \sum_{n\geq 0} c_n \frac{\zeta^n}{n!} \in \mathbb{C}[[\zeta]]$$

has positive radius of convergence and defines a holomorphic function of ζ which admits an analytic continuation along all the paths starting in its disc of convergence and lying in $\mathbb{C} \setminus \mathbb{Z}$ (see [1], or [7], or [8]§8). This property is stable by multiplication, the Borel transform of the Cauchy product $\widetilde{\varphi} \cdot \widetilde{\psi}$ being the convolution product $\widehat{\varphi} * \widehat{\psi}$ defined by

$$(\widehat{\varphi} * \widehat{\psi})(\zeta) = \int_0^\zeta \widehat{\varphi}(\zeta_1)\widehat{\psi}(\zeta - \zeta_1) \qquad \text{for } |\zeta| \text{ small enough} \tag{28}$$

(with $\widehat{\varphi}$ and $\widehat{\psi}$ denoting the Borel transforms of $\widetilde{\varphi}$ and $\widetilde{\psi}$ respectively).

A simple example is provided by the formal series $\widetilde{\mathcal{V}}^{\boldsymbol{\omega}}(z) = \mathcal{V}^{\boldsymbol{\omega}}(-1/z)$: indeed, the Borel transforms of the convergent series $\widetilde{a}_n(z) = a_n(-1/z)$ are entire functions $\widehat{a}_n(\zeta)$ and the Borel counterpart of $\partial = \frac{\mathrm{d}}{\mathrm{d}z}$ is multiplication by $-\zeta$, hence (19) yields

$$\widehat{\mathcal{V}}^{n_1}(\zeta) = -\frac{1}{\zeta - n_1}\widehat{a}_{n_1}(\zeta)$$

$$\widehat{\mathcal{V}}^{n_1,n_2}(\zeta) = \frac{1}{\zeta - (n_1+n_2)}\big(\widehat{a}_{n_1} * \widehat{\mathcal{V}}^{n_2}\big)$$

$$\vdots$$

$$\widehat{\mathcal{V}}^{n_1,\dots,n_r} = (-1)^r \frac{1}{\zeta - \hat{n}_1}\Big(\widehat{a}_{n_1} * \Big(\frac{1}{\zeta - \hat{n}_2}\Big(\widehat{a}_{n_2} * \Big(\cdots\Big(\frac{1}{\zeta - \hat{n}_r}\widehat{a}_{n_r}\Big)\cdots\Big)\Big)\Big)\Big)$$

with $\hat{n}_i = n_i + \cdots + n_r$. Since the $\widehat{a}_n$'s are entire functions of ζ, the function $\widehat{\mathcal{V}}^{n_1,\dots,n_r}$ is holomorphic with $\hat{n}_1,\dots,\hat{n}_r$ as only possible singularities.

Moreover, the above formula is sufficiently explicit to make it possible to give majorant series arguments so as to prove the uniform convergence of the series of holomorphic functions

$$\widehat{\varphi}_n = \sum_{n_1+\cdots+n_r+1=n} \beta_{\boldsymbol{\omega}} \widehat{\mathcal{V}}^{\boldsymbol{\omega}}, \qquad \widehat{\psi}_n = \sum_{n_1+\cdots+n_r+1=n} \beta_{\boldsymbol{\omega}} \widehat{\mathcal{V}'}^{\boldsymbol{\omega}}$$

(§8 of [8] is devoted to this task); in other words, *the formal series* φ_n *and* ψ_n *are resurgent with respect to* $z = -1/x$.

9 Conclusion

9.1. The reader is referred to [2] or [8] §9–11 for the resurgent approach to the question of the analytic classification of saddle-node singularities, which one can develop once the analytic structure of the functions $\widehat{\varphi}_n(\zeta)$ is clear. This approach relies on the use of Écalle's *alien calculus*: the singularities in the ζ-plane are controlled through operators Δ_m, $m \in \mathbb{Z}^*$, called alien derivations; mould calculus allows one to summarize the singular structure of the functions $\widehat{\mathcal{V}}^{\boldsymbol{\omega}}$ in the simple equation

$$\Delta_m \widetilde{\mathcal{V}}^\bullet = \widetilde{\mathcal{V}}^\bullet \times V^\bullet(m),$$

where $V^\bullet(m)$ is an alternal scalar-valued mould (the $V^{\boldsymbol{\omega}}$'s are complex numbers), with $n_1 + \cdots + n_r \neq m \;\Rightarrow\; V^{n_1,\dots,n_r}(m) = 0$.

This leads to Écalle's "Bridge Equation" for Θ, which gives the alien derivatives of all the resurgent functions $\widetilde{\varphi}_n$ and in which the resurgent solution of the analytic classification problem is subsumed.

9.2. Another application of mould calculus in Resurgence theory is the use of the formal series $\widetilde{\mathcal{V}}^{\boldsymbol{\omega}}$ to construct "resurgence monomials" $\widetilde{\mathcal{U}}^{\boldsymbol{\omega}}$ which behave as simply as possible under alien derivation. This allows one to prove that alien derivations generate an infinite-dimensional free Lie algebra.

9.3. Other problems of formal normalisation can be handled by an approach similar to the one explained in Section 4. See [8]§13 for the simple case of the linearisation of a vector field with non-resonant spectrum; much more complicated situations (taking into account resonances) are considered in [3] or [4]. The strategy always begins by expanding the object to study in a sum of homogeneous components B_n and considering the corresponding multiplicative comould $\mathbf{B}_\bullet$.

It must be mentioned that, when this is applied to the analysis of a local diffeomorphism, rather than a local vector field, the operators B_n are no longer derivations, but they still satisfy a kind of modified Leibniz rule:

$$\sigma(B_n) = B_n \otimes \mathrm{Id} + \sum_{n'+n''=n} B_{n'} \otimes B_{n''} + \mathrm{Id} \otimes B_n.$$

The resulting comould is not cosymmetr*a*l but cosymmetr*e*l, a property which involves "contracting shuffling coefficients" instead of the shuffling coefficients $\mathrm{sh}\binom{\boldsymbol{\omega}^1,\,\boldsymbol{\omega}^2}{\boldsymbol{\omega}}$. Correspondingly, the relevant symmetry properties for the moulds to be contracted into $\mathbf{B}_\bullet$ are "symmetrelity" and "alternelity".

The theory can thus be extended so as to treat on an equal footing vector fields and diffeomorphisms.

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